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Unsteady Newton-Busemann Flow Theory—Part I: Airfoils

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Newtonian flow theory for unsteady flow at very high Mach numbers is completed by the addition of a centrifugal force correction to the impact pressures. The correction term is the unsteady counterpart of Busemann's centrifugal force correction to impact pressures in steady flow. For airfoils of arbitrary shape, exact formulas for the unsteady pressure and stiffness and damping-in-pitch derivatives are obtained in closed form, which require only numerical quadratures of terms involving the airfoil shape. They are applicable to airfoils of arbitrary thickness having sharp or blunt leading edges. For wedges and thin airfoils these formulas are greatly simplified, and it is proved that the pitching motions of thin airfoils of convex shape and of wedges of arbitrary thickness are always dynamically stable according to Newton-Busemann theory. Leading-edge bluntness is shown to have a favorable effect on the dynamic stability; on the other hand, airfoils of concave shape tend toward dynamic instability over a range of axis positions if the surface curvature exceeds a certain limit. As a byproduct, it is also shown that a pressure formula recently given by Barron and Mandl for unsteady Newtonian flow over a pitching power-law shaped airfoil is erroneous and that their conclusion regarding the effect of pivot position on the dynamic stability is misleading.

I. Introduction

IN the Newtonian model for fluid flow,[1] fluid particles do not interact with each other; however, upon striking a body, they lose their normal component of velocity relative to the body surface. The particles then continue along the frictionless surface with zero tangential acceleration. This change in the normal component of velocity before and after impact (Δv_n) yields a pressure

$$p_{\text{impact}} = \rho_\infty (\Delta v_n)^2 \tag{1}$$

where ρ_∞ is the fluid density in the freestream.

It was first pointed out by Busemann[2] that, for bodies with curved surfaces, the pressure given by Eq. (1) represents that at the top of the Newtonian shock layer, but the pressure at the body surface must contain additional centrifugal corrections due to the curved trajectories followed by the fluid particles. Thus,

$$p_{\text{surface}} = p_{\text{impact}} + p_{\text{centrif}} \tag{2}$$

For steady flow, the rational relation between this Newton-Busemann theory and the modern gasdynamic theory was first established by Cole[3] for slender axisymmetric bodies and later by Hui[4] for general two-dimensional and axisymmetric bodies. In particular, it was shown in Ref. 4 that the Newton-Busemann pressure, Eq. (2), can be obtained from gasdynamic theory in the double limit as the ratio of the specific heats of the gas γ approaches unity and independently as the freestream Mach number M_∞ approaches infinity.

The centrifugal pressure correction p_{centrif} is evidently directly dependent on the curvature of the particles' trajectories. In the case of steady flow, since these trajectories are coincident with the geodesics of the body surface, the calculation of their curvature and hence of p_{centrif} is relatively straightforward. This, unfortunately, is not the case for unsteady flows. Although the fluid particles in the unsteady flow case are still assumed to follow the body surface, and hence their trajectories must be tangential to the surface at all times, the curvature of these trajectories is not known in general, as it is not the same as the instantaneous curvature of the body surface but is additionally dependent on the motion of the body that causes the flow unsteadiness. This is the first difficulty that has to be overcome in calculating p_{centrif} in unsteady flows. Another difficulty lies in applying the law of conservation of mass in unsteady motions.

Recently, Mahood and Hui[5] studied the two special cases of unsteady flow past oscillating wedges and cones at zero-mean incidence using Newton-Busemann theory. In both cases, although the steady centrifugal force corrections are zero, the unsteady parts are not. Mahood and Hui obtained these unsteady corrections and showed that when they are included, the Newton-Busemann pressure agrees identically with the gasdynamic solutions for oscillating wedges[6] and for oscillating cones[7] in the double limit $\gamma \to 1$ and $M_\infty \to \infty$, independently. It was also conjectured that the equivalency of Newton-Busemann theory and the limiting gasdynamic theory in unsteady flow holds for bodies of general shape.

More recently, Barron and Mandl[8] used limiting gasdynamic theory to calculate the unsteady Newtonian flow relevant to pitching oscillations of power-law shaped airfoils (with the power m restricted to the range $2/3 < m < 4/5$). The unsteady surface pressure that results from their analysis is independent of the pivot position h, yielding a damping-in-pitch derivative that is linearly, instead of quadratically, dependent on h. They thus concluded that whether or not the pitching motion of these airfoils is dynamically stable depends on whether h is greater or smaller than a number, $(3m-1)^2/(9m^2)$. It will be shown later in this paper that the unsteady Newtonian pressure given by Barron and Mandl is erroneous and that their conclusion regarding dynamic stability is misleading.

The purpose of the present paper is to develop a general method for calculating the Newton-Busemann surface pressure relevant to the pitching and plunging oscillations of airfoils of arbitrary thickness and shape having sharp or blunt leading edges. In Sec. II the Newtonian impact pressure will first be derived, and in Sec. III the trajectories of the particles will be determined. These will be used in finding the steady and unsteady centrifugal force corrections in Sec. IV. In Sec. V these general results will be applied to study the dynamic stability of classes of airfoils and, finally, conclusions will be given in Sec. VI.

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II. Newtonian Impact Pressure

Consider an airfoil performing a harmonic pitching oscillation about the pivot axis C in a uniform oncoming flow U_∞. Simultaneously, the pivot axis undergoes a harmonic plunging oscillation so that the airfoil is in a combined pitching and plunging motion (Fig. 1). Let a flow-fixed system of Cartesian coordinates $X0Y$ be such that 0 is at the mean position of the leading edge and $0X$ is along the direction of the freestream U_∞. Let a body-fixed system of Cartesian coordinates $x0'y$ be such that $0'$ is at the leading edge and $0'x$ is directed along the chord line of the airfoil. In the following, all the lengths, X, x, Y, y, h, etc., are scaled by the length of the airfoil l, velocities v by U_∞, density ρ by ρ_∞, pressure p by $\rho_\infty U_\infty^2$ and the time value t by l/U_∞. Thus, for example, $\rho_\infty = U_\infty = l = 1$.

The harmonic pitching oscillation of the airfoil with frequency ω may be represented by the displacement angle θ (i.e.., the angle of pitch, Fig. 1)

$$\theta(t) = \bar{\theta} e^{ikt} \tag{3}$$

where

$$k = \omega l / U_\infty \tag{4}$$

is the reduced frequency and $\bar{\theta}$ is the amplitude of oscillation. Likewise, the harmonic plunging oscillation of the airfoil at the same frequency ω may be represented by the linear displacement Y_c (Fig. 1) where

$$Y_c(t) = \bar{Y}_c e^{ikt} \tag{5}$$

To conform with notation familiar to workers in aerodynamic stability, we call the angle formed by the ratio $\dot{Y}_c / U_\infty$ the flight-path angle γ; i.e.,

$$\dot{Y}_c(t) = ik\bar{Y}_c e^{ikt} \equiv \gamma(t) = \bar{\gamma} e^{ikt} \tag{6}$$

Letting the airfoil undergo variations in both $\theta(t)$ and $\gamma(t)$ will later enable us to evaluate not only the aerodynamic damping-in-pitch coefficient $C_{m_{\dot\theta}}$ due to pitching oscillations in rectilinear flight ($\gamma = 0$), but also the separate contributions to it—C_{m_q} (due to purely pitching variations) and $C_{m_{\dot\alpha}}$ (due to purely plunging variations). It will be assumed that $|\bar{\theta}|$, $|\bar{\gamma}|$, and k are all $\ll 1$, and all terms of $O(\bar{\theta}^2, \bar{\gamma}^2, \bar{\theta}k^2, \bar{\gamma}k^2)$ and higher will be neglected.

Let us represent a point on the airfoil surface by the vector sum of two position vectors r_0 and r_1 (Fig. 1): r_0 defines the position of the pivot axis C relative to the flow-fixed system $X0Y$, while r_1 defines the point on the airfoil surface relative to C, a point fixed in the airfoil. Thus, r_1 can be defined directly in terms of the body-axis coordinates. In the latter system, let the equation of the airfoil surface be given by

$$y = f(x) \tag{7}$$

where, evidently, $f(0) = 0$. Then the position vector of a point P on the airfoil surface can be written as

$$r = r_0 + r_l = iX + jY \tag{8}$$

Here,

$$X = X_c + (x-h)\cos\theta - f(x)\sin\theta$$

$$Y = Y_c(t) + (x-h)\sin\theta + f(x)\cos\theta \tag{9}$$

where X_c is a constant equal in magnitude to h, which in the body-axis system is the chordwise distance of the pivot axis C from the nose. The unit tangential vector τ and the unit normal vector n at the point P are then given by

$$\tau = i\cos\epsilon + j\sin\epsilon$$

$$n = -i\sin\epsilon + j\cos\epsilon \tag{10}$$

where ϵ is the inclination angle of the tangential direction τ relative to the flow-fixed system, and is given by

$$\epsilon = \theta + \tan^{-1} f'(x) \tag{11}$$

Then, for $\bar{\theta} \ll 1$,

$$\sin\epsilon = \mu(x)[\theta(t) + f'(x)]$$

$$\cos\epsilon = \mu(x)[1 - \theta(t) f'(x)] \tag{12}$$

where

$$\mu(x) = [1 + f'^2(x)]^{-1/2} \tag{13}$$

Now the normal component of velocity before impact is

$$(v_n)_{\text{before}} = -\sin\epsilon = -\mu(x)[\theta(t) + f'(x)] \tag{14}$$

After impact, a fluid particle moves along the surface. The normal component of its velocity at a point x is the same as that of the surface itself. Thus,

$$(v_n)_{\text{after}} = \dot{X}\sin\epsilon + \dot{Y}\cos\epsilon$$
$$= \mu(x)\{\dot{\theta}(t)[x - h + f(x) f'(x)] + \gamma(t)\} \tag{15}$$

Accordingly, the pressure due to particle impact as given by Eq. (1) is:

$$p(x,t)_{\text{impact}} = \mu^2(x)\{f'^2(x) + 2f'(x)[\theta(t) + \gamma(t)]$$
$$+ 2f'(x)\dot{\theta}(t)[x - h + f(x) f'(x)]\} \tag{16}$$

[( )′ denotes ordinary derivative and ($\dot{}$) = d/dt.]

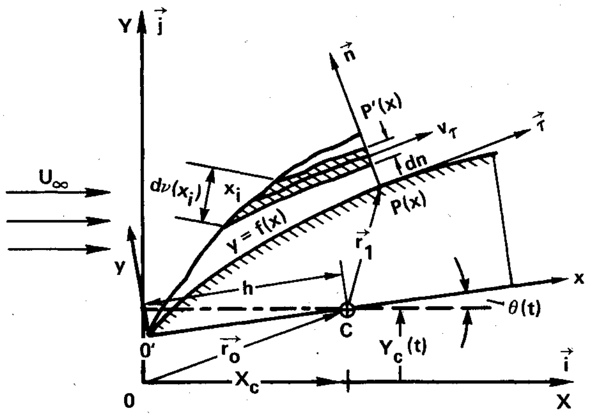

Fig. 1 Oscillating airfoil showing structure of infinitesimal Newtonian shock layer.

III. Particle Trajectories

Calculation of the additional surface pressure due to centrifugal effects requires knowledge of the trajectories of the fluid particles. A study of the trajectories will be undertaken in this section.

According to the assumption of Newtonian flow theory, once a particle strikes the surface, its subsequent motion will be along the surface. Its position vector may then be written as

$$r = iX + jY \tag{17}$$

where, for $\bar{\theta} \ll 1$,

$$X = X_c + (x-h) - \theta(t) f(x)$$
$$Y = Y_c(t) + \theta(t)(x-h) + f(x) \tag{18}$$

The velocity and acceleration vectors of the particle are, of course, given by $v = dr/dt$ and $a = dv/dt$. Note here that since x defines the instantaneous postion of a particle relative to the body-axis system, the expression for velocity and acceleration must include time derivatives of x and, likewise, $f(x)$; i.e., $d/dt[f(x)] = f'(x)\dot{x}$. In particular, we obtain the following for the tangential velocity component v_τ, normal component of acceleration a_n, and tangential component of acceleration a_τ:

$$v_\tau = \mu\{(\dot{x}/\mu^2) + \dot{\theta}[(x-h)f'(x) - f(x)] + \gamma f'(x)\} \tag{19}$$

$$a_n = \mu[\dot{x}^2 f''(x) + (2\dot{\theta}/\mu^2)\dot{x} + \dot{\gamma}] \tag{20}$$

$$a_\tau = \mu[(\ddot{x}/\mu^2) + \dot{x}^2 f'(x) f''(x) + \dot{\gamma} f'(x)] \tag{21}$$

The trajectory of the particle after impact is determined by the basic assumption of Newtonian flow that the tangential component of acceleration be zero. Thus, from Eq. (21), we get

$$\ddot{x} + \mu^2(x) f'(x) f''(x) \dot{x}^2 + \dot{\gamma}\mu^2(x) f'(x) = 0 \tag{22}$$

Equation (22) can be cast in a much simpler form if we note in Eq. (19) that v_τ, the tangential component of velocity, is composed of two distinct parts: one, involving $\dot{x}$, which is the tangential velocity component of the particle measured relative to the airfoil, and the other, involving $\dot{\theta}$ and γ, which arises from the velocity of the airfoil itself relative to the flow-fixed frame XOY. Calling u the component of tangential velocity relative to the body-axis system, we have from Eq. (19),

$$u = \dot{x}/\mu \tag{23}$$

so that

$$\dot{u} = (\ddot{x}/\mu) - (\mu'/\mu^2)\dot{x}^2 \tag{24}$$

Noting that $\mu' = -\mu^3 f' f''$, we see that Eq. (22) is equivalently

$$\dot{u} + \dot{\gamma}\mu f' = 0 \tag{25}$$

Now Eq. (25) is an expression for the motion of a particular particle as it enters and traverses the shock layer. As a boundary condition on Eq. (25), we know that immediately after impact at a station x_i, the tangential component of relative velocity u is composed of the component of the uniform freestream in the tangential direction plus components due to the motion of the airfoil, which will appear to be oncoming flow velocities to an observer in the body-fixed axis system. The value of u at x_i is found to be

$$u(x_i,t) = \mu(x_i)\{1 - f'(x_i)[\theta(t) + \gamma(t)]$$
$$-\dot{\theta}(t)[(x_i - h)f'(x_i) - f(x_i)]\} \tag{26}$$

Notice that Eq. (26) is compatible with Eq. (19) when the left-hand side of Eq. (19) is identified with the component of the oncoming freestream, $\cos\epsilon(x_i) \approx \mu(x_i)[1 - \theta(t) f'(x_i)]$.

A solution of Eq. (25), which is valid for small reduced frequency k and is compatible with Eq. (26), can be obtained in the form

$$u = A(x) + \theta(t)B(x) + \dot{\theta}(t)C(x)$$
$$+\gamma(t)D(x) + \dot{\gamma}(t)E(x) \tag{27}$$

Substituting Eq. (27) into Eqs. (25) and (26), using the relation

$$\frac{dQ(x)}{dt} = Q'(x)\dot{x} = Q'(x)\mu u$$

$$= Q'\mu[A + \theta B + \dot{\theta}C + \gamma D + \dot{\gamma}E] \tag{28}$$

and equating like terms in θ, $\dot{\theta}$, γ, $\dot{\gamma}$, etc., we obtain the following simple set of problems:

$$A(x)A'(x) = 0, \qquad A(x_i) = \mu(x_i) \tag{29}$$

$$AB' + A'B = 0, \qquad B(x_i) = -\mu(x_i)f'(x_i) \tag{30}$$

$$B + \mu(AC' + A'C) = 0$$

$$C(x_i) = -\mu(x_i)[(x_i - h)f'(x_i) - f(x_i)] \tag{31}$$

$$AD' + A'D = 0, \qquad D(x_i) = -\mu(x_i)f'(x_i) \tag{32}$$

$$D + \mu(AE' + A'E) + \mu f' = 0, \qquad E(x_i) = 0 \tag{33}$$

Solving the set, Eqs. (29-33), consecutively leads to the following expression for u:

$$u(x,x_i,t) = \mu(x_i)\{1 - f'(x_i)[\theta(t) + \gamma(t)]$$
$$-\dot{\theta}(t)[(x_i - h)f'(x_i) - f(x_i)]\}$$
$$+[\dot{\theta}(t) + \dot{\gamma}(t)]f'(x_i)\int_{x_i}^{x}\frac{d\xi}{\mu(\xi)} - \frac{\dot{\gamma}(t)}{\mu(x_i)}[f(x) - f(x_i)] \tag{34}$$

IV. Centrifugal Force Correction

We are now in a position to calculate the centrifugal correction to the surface pressure due to the curved trajectories followed by the particles. Thus, consider such a correction at a point $P(x)$ on the surface at time t. In Fig. 1 the Newtonian shock layer $0'PP'$ is shown magnified, where $0'P'$ is the outer edge of the layer. The layer is actually of infinitesimal thickness, and the distance PP' collapses.

The contribution to centrifugal pressure from the element of constant thickness dn (the shaded region in Fig. 1) is given by

$$\frac{-1}{\rho(x,x_i,t)}\frac{dp}{dn}(x,x_i,t) = a_n(x,x_i,t) \tag{35}$$

The total centrifugal correction to pressure across the shock layer from the outer edge P' to P at the surface is

$$p_{\text{centrif}} = \int_{P'}^{P}\frac{dp}{dn}dn = \int_{P}^{P'} a_n \rho dn \tag{36}$$

Having an expression for a_n from Eq. (20), we need one for ρdn, which we obtain from consideration of the law of conservation of mass within the shock layer. In a frame of reference fixed with respect to the body, application of the law to the element of constant thickness dn gives

$$\frac{\partial}{\partial t}(\rho dn) + \nabla\cdot(\boldsymbol{u}\rho dn) = 0 \tag{37}$$

where $\boldsymbol{u}$ is the particle velocity relative to the airfoil. Since $\boldsymbol{u}$ has only one component u along the tangential direction, Eq.

(37) reduces to

$$\frac{\partial}{\partial t}(\rho \mathrm{d}n) + \mu \frac{\partial}{\partial x}(u\rho \mathrm{d}n) = 0 \tag{38}$$

The boundary condition on ρdn is obtained from the requirement that the mass flow into the element be continuous across the boundary of the shock layer at $x = x_i$. The mass flow rate outside the shock layer has the form (note: $\rho_\infty = 1$)

$$\mathrm{d}\dot{m}(x_i,t) = [1 + \dot{\theta}(t) f(x_i)]\mathrm{d}\nu(x_i,t) \tag{39}$$

where the factor $[1 + \dot{\theta}(t) f(x_i)]$ is the magnitude of the oncoming relative flow velocity at x_i, and $d\nu(x_i,t)$ is taken normal to the direction of that flow (Fig. 1). A simple calculation gives for $d\nu(x_i,t)$

$$\mathrm{d}\nu(x_i,t) = [f'(x_i) + \theta(t) + \gamma(t) + \dot{\theta}(t)(x_i - h)]\mathrm{d}x_i \tag{40}$$

Within the shock layer at x_i, the mass flow rate is

$$\mathrm{d}\dot{m}(x_i,t) = \rho(x_i,t) u(x_i,t)\mathrm{d}n \tag{41}$$

Equating Eqs. (39) and (41), and introducing Eqs. (26) and (40) for u and $d\nu$, we get for the boundary condition on ρdn at x_i:

$$\rho \mathrm{d}n(x_i,t) = \frac{\mathrm{d}x_i}{\mu(x_i)}\left\{f'(x_i) + \frac{1}{\mu^2(x_i)}[\theta(t) + \gamma(t)] + \frac{\dot{\theta}(t)}{\mu^2(x_i)}(x_i - h)\right\} \tag{42}$$

A solution of Eq. (38) for ρdn, compatible with Eq. (42), can be obtained in the same way that the solution for u was obtained. Let

$$\rho dn = \hat{A}(x) + \theta(t)\hat{B}(x) + \dot{\theta}(t)\hat{C}(x) + \gamma(t)\hat{D}(x) + \dot{\gamma}(t)\hat{E}(x) \tag{43}$$

so that, to within terms of $O(\ddot{\theta}, \ddot{\gamma})$,

$$(\partial/\partial t)(\rho \mathrm{d}n) = \dot{\theta}\hat{B}(x) + \dot{\gamma}\hat{D}(x) \tag{44}$$

Substituting Eqs. (43), (44), and (27) in Eq. (38), matching first-order terms in the angular variables, and selecting the appropriate parts of the boundary condition Eq. (42), we get the following set of problems for the coefficients $\hat{A}, \ldots, \hat{E}(x)$:

$$\frac{\partial}{\partial x}(A\hat{A}) = 0, \qquad \hat{A}(x_i) = \frac{f'(x_i)}{\mu(x_i)}\mathrm{d}x_i \tag{45a}$$

$$\frac{\partial}{\partial x}(A\hat{B} + \hat{A}B) = 0, \qquad \hat{B}(x_i) = \frac{\mathrm{d}x_i}{\mu^3(x_i)} \tag{45b}$$

$$\hat{B} + \mu(x)\frac{\partial}{\partial x}(A\hat{C} + \hat{A}C) = 0, \qquad \hat{C}(x_i) = \frac{(x_i - h)}{\mu^3(x_i)}\mathrm{d}x_i \tag{45c}$$

$$\frac{\partial}{\partial x}(A\hat{D} + \hat{A}D) = 0, \qquad \hat{D}(x_i) = \frac{\mathrm{d}x_i}{\mu^3(x_i)} \tag{45d}$$

$$\hat{D} + \mu(x)\frac{\partial}{\partial x}(A\hat{E} + \hat{A}E) = 0, \qquad \hat{E}(x_i) = 0 \tag{45e}$$

Solving the set Eq. (45) gives for ρdn within the shock layer,

$$\rho \mathrm{d}n(x,x_i,t) = \frac{\mathrm{d}x_i}{\mu(x_i)}\Bigg(f'(x_i) + [\theta(t) + \gamma(t)]\frac{1}{\mu^2(x_i)} + \dot{\theta}(t)\left\{\frac{(x_i - h)}{\mu^2(x_i)} - \frac{[1 + \mu^2(x_i)f'^2(x_i)]}{\mu^3(x_i)}\int_{x_i}^{x}\frac{\mathrm{d}\xi}{\mu}\right\} + \dot{\gamma}(t)\left\{\frac{f'(x_i)}{\mu^2(x_i)}[f(x) - f(x_i)] - \frac{[1 + \mu^2(x_i)f'^2(x_i)]}{\mu^3(x_i)}\int_{x_i}^{x}\frac{\mathrm{d}\xi}{\mu}\right\}\Bigg) \tag{46}$$

Finally, returning to Eq. (20) and using Eq. (23), we recast the expression for a_n in the form

$$a_n(x,x_i,t) = \kappa(x)u^2(x,x_i,t) + 2\dot{\theta}(t)u(x,x_i,t) + \mu(x)\dot{\gamma}(t) \tag{47}$$

where we recognize

$$\kappa(x) = \mu^3(x)f''(x) \tag{48}$$

as being the curvature of the airfoil surface. Substituting Eqs. (46) and (47) in Eq. (36) and using Eq. (34) for particle velocity u, we note that the summation of the elements dn across the shock layer is now represented by the summation of elements dx_i over the interval $0 < x_i < x$. The result for the centrifugal pressure correction takes the form

$$\begin{aligned} p_{centrif}(x,t) &= \kappa(x)\int_0^x \mu(x_i)f'(x_i)\mathrm{d}x_i \\ &+ [\theta(t) + \gamma(t)]\kappa(x)\int_0^x \mu(x_i)[1 - f'^2(x_i)]\mathrm{d}x_i \\ &+ \dot{\theta}(t)\Big(2f(x) + \kappa(x)\int_0^x \mu(x_i)\Big\{(x_i - h)[1 - f'^2(x_i)] \\ &+ 2f(x_i)f'(x_i)\Big\}\mathrm{d}x_i - \kappa(x)\int_0^x \mathrm{d}x_i\int_{x_i}^x \frac{\mathrm{d}\xi}{\mu(\xi)}\Big) \\ &+ \dot{\gamma}(t)\Big\{\mu(x)\int_0^x \frac{f'(x_i)}{\mu(x_i)}\mathrm{d}x_i \\ &- \kappa(x)\int_0^x \frac{f'(x_i)}{\mu(x_i)}[f(x) - f(x_i)]\mathrm{d}x_i \\ &- \kappa(x)\int_0^x \mathrm{d}x_i\int_{x_i}^x \frac{\mathrm{d}\xi}{\mu(\xi)}\Big\} \end{aligned} \tag{49}$$

Equation (49) together with Eq. (16) gives the complete surface pressure according to Newton-Busemann theory. The complete result may be put in the convenient form

$$p(x,t) = P_0(x) + \theta(t)P_1(x) + \dot{\theta}(t)[P_2(x) - hP_1(x)] + \gamma(t)P_1(x) + \dot{\gamma}(t)P_3(x) \tag{50}$$

where the coefficients $P_0(x), \ldots, P_3(x)$ are given by

$$P_0(x) = \mu^2(x)f'^2(x) + \kappa(x)\int_0^x \mu(\xi)f'(\xi)\mathrm{d}\xi \tag{51a}$$

$$P_1(x) = 2\mu^2(x)f'(x) + \kappa(x)\int_0^x \mu(\xi)[1 - f'^2(\xi)]\mathrm{d}\xi \tag{51b}$$

$$P_2(x) = 2\mu^2(x)f'(x)[x+f(x)f'(x)] + 2f(x)$$
$$+\kappa(x)\int_0^x \mu(\xi)\left\{\xi[1-f'^2(\xi)] + 2f(\xi)f'(\xi)\right\}d\xi$$
$$-\kappa(x)\int_0^x d\xi\int_\xi^x \frac{ds}{\mu(s)} \tag{51c}$$

$$P_3(x) = \mu(x)\int_0^x \frac{f'(\xi)}{\mu(\xi)}d\xi - \kappa(x)\int_0^x \frac{f'(\xi)}{\mu(\xi)}[f(x)-f(\xi)]d\xi$$
$$-\kappa(x)\int_0^x d\xi\int_\xi^x \frac{ds}{\mu(s)} \tag{51d}$$

We see that the coefficients $P_0(x),\ldots,P_3(x)$ are determined by quadratures depending only on the body shape which may be prescribed arbitrarily.

We point out here that $P_0(x)$ in Eqs. (51) is the well-known Newton-Busemann formula for pressure on the surface of a two-dimensional body in steady flow, whereas $P_1(x)$, $P_2(x)$, and $P_3(x)$ are new results. It is also important to note that while $P_2(x)$ and $P_3(x)$ are genuinely unsteady flow properties, $P_1(x)$, the coefficient of θ in Eq. (50), is derivable from steady flow considerations alone. In Appendix A, we make use of this fact to derive $P_1(x)$ by an alternative method utilizing the Newton-Busemann formula for $P_0(x)$. Now it is proved in Ref. 4 that results from steady two-dimensional Newton-Busemann theory must be the same as those from gasdynamic theory in the limit as Mach number tends to infinity, and independently as the ratio of specific heats tends to unity. It follows, therefore, that our result for $P_1(x)$ in Eqs. (51) must be identical to the result that would be obtained from gasdynamic theory. Conversely, this means that to the first order in frequency, a correct limiting gasdynamic theory must yield an equation for pressure in which the coefficient of θ is identical to $P_1(x)$ in Eqs. (51). We note in passing that the formula derived from limiting gasdynamic theory by Barron and Mandl[8] for the relevant (in-phase) part of the unsteady pressure [Re P in Eq. (27) of Ref. 8] on power-law shaped bodies ($f(x)=bx^m$, $2/3<m<4/5$), when expressed in body-fixed coordinates, fails to meet this test (see the Appendix). Since the result for Re P also figures in the derivation of the out-of-phase part of the unsteady pressure [Im P in Eq. (27) of Ref. 8], that result must also be incorrect. In fact, since according to Ref. 8, Re P vanishes identically for steady flow, it is probably a consequence of this error that Im P fails to show any dependence on axis position h.

V. Stability of Oscillating Airfoils

It is customary in stability studies to consider that the principal motion variables are angle of attack $\alpha(t)$ and angle of pitch $\theta(t)$. We note that $\alpha(t)$, the angle between the chord line of the airfoil and the oncoming uniform relative wind, is given by the sum of $\theta(t)$ and $\gamma(t)$. Introducing $\alpha(t)$ in place of $\gamma(t)$, we rewrite the expression for surface pressure, Eq. (50), as

$$p(x,t) = P_0(x) + \alpha(t)P_1(x)$$
$$+\dot\theta(t)[P_2(x) - P_3(x) - hP_1(x)] + \dot\alpha(t)P_3(x) \tag{52}$$

The normal force and pitching-moment coefficients are introduced, defined as usual by

$$C_N = \frac{N}{½\rho_\infty U_\infty^2} \qquad C_m = \frac{M}{½\rho_\infty U_\infty^2 l} \tag{53}$$

where N is the normal force and M is the moment of surface-pressure force about the pivot axis C. For simplicity, we consider symmetric airfoils at zero-mean angle of attack, and write

$$C_N = \dot\theta(t)C_{N_{\dot\theta}} + \alpha(t)C_{N_\alpha} + \dot\alpha(t)C_{N_{\dot\alpha}} + 0(\bar\theta^2,\bar\alpha^2,\bar\theta k^2,\bar\alpha k^2)$$
$$C_m = \dot\theta(t)(-C_{m_{\dot\theta}}) + \alpha(t)(-C_{m_\alpha}) + \dot\alpha(t)(-C_{m_{\dot\alpha}})$$
$$+0(\bar\theta^2,\bar\alpha^2,\bar\theta k^2,\bar\alpha k^2) \tag{54}$$

where the coefficients have the form‡

$$C_{N_\phi} = 4\int_0^1 \frac{\partial p(x,t)}{\partial\phi}dx, \qquad \phi = \dot\theta,\alpha,\dot\alpha$$
$$-C_{m_\phi} = 4\int_0^1 \frac{\partial p(x,t)}{\partial\phi}[x-h+f(x)f'(x)]dx, \qquad \phi=\dot\theta,\alpha,\dot\alpha \tag{55}$$

The following three cases will serve to define the pertinent stability derivatives: 1) motions involving purely angle-of-attack variations, $\alpha=\alpha(t)$, $\theta=0$; 2) motions involving purely pitching variations, $\theta=\theta(t)$, $\alpha=0$; 3) pitching motion in rectilinear flight ($\gamma=0$), $\theta(t)=\alpha(t)$. We shall give the form of the stability derivatives that these motions define.

A. Purely Angle-of-Attack Variations, $\alpha=\alpha(t)$, $\theta=0$

Substituting Eq. (52) (with $\dot\theta=0$) in Eq. (55) and matching terms in Eq. (54), we get for the stability derivatives

$$C_{N_\alpha} = 4\int_0^1 P_1(x)dx, \qquad C_{N_{\dot\alpha}} = 4\int_0^1 P_3(x)dx$$
$$-C_{m_\alpha} = 4\left\{\int_0^1 P_1(x)[x+f(x)f'(x)]dx - h\int_0^1 P_1(x)dx\right\}$$
$$-C_{m_{\dot\alpha}} = 4\left\{\int_0^1 P_3(x)[x+f(x)f'(x)]dx - h\int_0^1 P_3(x)dx\right\} \tag{56}$$

B. Purely Pitching Variations, $\theta=\theta(t)$, $\alpha=0$

To distinguish the stability derivatives corresponding to purely pitching velocity variations from those of the next case, where $\dot\theta=\dot\alpha$, we conform to common practice by letting $\dot\theta(t)\mid_{\alpha=0}\equiv q(t)$ in the present case. The results are

$$C_{N_q} = 4\left\{\int_0^1 [P_2(x)-P_3(x)]dx - h\int_0^1 P_1(x)dx\right\}$$
$$-C_{m_q} = 4\left\{\int_0^1 [P_2(x)-P_3(x)][x+f(x)f'(x)]dx\right.$$
$$-h\int_0^1 P_1(x)[x+f(x)f'(x)]dx$$
$$\left.-h\int_0^1 [P_2(x)-P_3(x)]dx + h^2\int_0^1 P_1(x)dx\right\} \tag{57}$$

C. Pitching Motion in Rectilinear Flight ($\gamma=0$), $\theta(t)=\alpha(t)$

Letting the angle variable be defined by θ, we see from Eq. (54) that the normal-force and pitching-moment stability derivative coefficients are simply the sums of the corresponding separate coefficients due to purely pitching and

‡The indication of partial derivatives with respect to $\dot\theta$, α, $\dot\alpha$ in Eq. (54) is not strictly correct, since, e.g., α and $\dot\alpha$ are not independent quantities. However, the assumption at this point that they are independent quantities is a convenient fiction, the adoption of which can be justified by the knowledge that a more rigorous approach will yield the same end result.

purely angle-of-attack variations. The results are

$$C_{N_\theta} = C_{N_\alpha} = 4\int_0^1 P_1(x)\,dx$$

$$C_{N_{\dot\theta}} = (C_{N_q} + C_{N_{\dot\alpha}}) = 4\left[\int_0^1 P_2(x)\,dx - h\int_0^1 P_1(x)\,dx\right]$$

$$-C_{m_\theta} = C_{m_\alpha} = 4\left\{\int_0^1 P_1(x)\,[x + f(x)f'(x)]\,dx - h\int_0^1 P_1(x)\,dx\right\}$$

$$-C_{m_{\dot\theta}} = -(C_{m_q} + C_{m_{\dot\alpha}}) = 4\left\{\int_0^1 P_2(x)\,[x+f(x)f'(x)]\,dx - h\int_0^1 P_1(x)\,[x+f(x)f'(x)]\,dx - h\int_0^1 P_2(x)\,dx + h^2\int_0^1 P_1(x)\,dx\right\} \quad (58)$$

We note additionally from Eqs. (56) and (57) that the terms C_{N_q}, $-C_{m_q}$, and $-C_{m_{\dot\alpha}}$ in Eq. (58) individually obey the axis-transfer rules that have been found by previous authors to apply generally to all classes of symmetric bodies. The complete set of transfer rules that the coefficients obey is given for reference below.

$$-C_{m_\alpha} = -C_{m_{\alpha_0}} - hC_{N_\alpha}, \qquad -C_{m_{\dot\alpha}} = -C_{m_{\dot\alpha_0}} - hC_{N_{\dot\alpha}},$$

$$C_{N_q} = C_{N_{q_0}} - hC_{N_\alpha}$$

$$-C_{m_q} = -C_{m_{q_0}} - hC_{N_{q_0}} - h(-C_{m_{\alpha_0}}) + h^2 C_{N_\alpha}$$

$$C_{N_{\dot\theta}} = C_{N_{\dot\theta_0}} - hC_{N_\alpha} = (C_{N_{q_0}} + C_{N_{\dot\alpha}}) - hC_{N_\alpha}$$

$$-C_{m_{\dot\theta}} = -C_{m_{\dot\theta_0}} - hC_{N_{\dot\theta_0}} - h(-C_{m_{\alpha_0}}) + h^2 C_{N_\alpha}$$

$$= -(C_{m_{q_0}} + C_{m_{\dot\alpha_0}}) - h(C_{N_{q_0}} + C_{N_{\dot\alpha}}) - h(-C_{m_{\alpha_0}}) + h^2 C_{N_\alpha} \quad (59)$$

Here, h is the axis position, measured positive rearward from the nose, and the subscripted terms are the stability derivatives as calculated for $h=0$.

With reference to the results for the stiffness derivative $-C_{m_\theta}$ and the damping-in-pitch derivative $-C_{m_{\dot\theta}}$ arising from pitching oscillations in rectilinear flight (Eq. 58), we see that $-C_{m_\theta}$ is linear in h while $-C_{m_{\dot\theta}}$ is quadratic in h. The latter result contradicts and renders inappropriate the principal result of Barron and Mandl[8] who, on the basis of their analysis, concluded that the damping-in-pitch derivative of power-law bodies varies only linearly with h, implying (incorrectly) that these bodies should experience dynamic instability for all values of h below a certain particular value.

Using Eq. (58) as a basis, we shall now study in detail the static and dynamic stability of two particular classes of airfoils oscillating in pitch in rectilinear flight.

D. Wedges

For a wedge

$$f(x) = x\tan\epsilon \quad (60)$$

where the constant ϵ is the semivertex angle of the wedge. From Eqs. (50) and (51) (with $\gamma, \dot\gamma = 0$), we get

$$P_0(x) = \sin^2\epsilon, \quad P_1(x) = 2\sin\epsilon\cos\epsilon, \quad P_2(x) = 4x\tan\epsilon \quad (61)$$

whence, From Eq. (58)

$$-C_{m_\theta} = 4\tan\epsilon(1 - 2h\cos^2\epsilon)$$

$$-C_{m_{\dot\theta}} = 4\frac{\tan\epsilon}{\cos^2\epsilon}\left[\frac{4}{3} - 3h\cos^2\epsilon + 2(h\cos^2\epsilon)^2\right] \quad (62)$$

Equations (62) for the stability derivatives of a wedge oscillating in pitch generalize the results of Mahood and Hui[5] to include the dependence on pivot position and are identical with the exact gasdynamic results of Hui[6] in the double limit as, independently, the ratio of specific heats γ approaches unity and the Mach number M_∞ approaches infinity. It is of interest to note that the damping-in-pitch derivative $-C_{m_{\dot\theta}}$ is always positive, from which we conclude that the pitching motion of a wedge of any thickness and about any axis position is dynamically stable in the Newtonian limit. This is to be compared with the finding of Hui[6] for finite Mach numbers which shows that the pitching motion of a wedge may become dynamically unstable if the Mach number is low enough or if the wedge is thick enough.

E. Thin Airfoils

For thin airfoils of arbitrary shape, let the airfoil thickness be characterized by b ($b \ll 1$) so that

$$f(x) = 0(b) \quad (63)$$

When terms of $0(b^3)$ are neglected, we have from Eqs. (50) and (51) (again with γ, $\dot\gamma = 0$),

$$P_0(x) = f'^2(x) + f(x)f''(x) = \frac{\partial}{\partial x}(ff')$$

$$P_1(x) = 2f'(x) + xf''(x) = \frac{\partial^2}{\partial x^2}(xf)$$

$$P_2(x) = 2xf'(x) + 2f(x) = 2\frac{\partial}{\partial x}(xf) \quad (64)$$

whence, from Eq. (58),

$$\frac{C_{N_\theta}}{4} = \int_0^1 P_1(x)\,dx = f(1) + f'(1)$$

$$\frac{-C_{m_{\theta_0}}}{4} = \int_0^1 xP_1(x)\,dx = f'(1)$$

$$\frac{C_{N_{\dot\theta_0}}}{4} = \int_0^1 P_2(x)\,dx = 2f(1)$$

$$\frac{-C_{m_{\dot\theta_0}}}{4} = \int_0^1 xP_2(x)\,dx = 2\left[f(1) - \int_0^1 xf(x)\,dx\right] \quad (65)$$

The stiffness and damping-in-pitch derivatives about an arbitrary axis can be formed from Eq. (65) [cf. Eqs. (58) and (59)] to give

$$\frac{-C_{m_\theta}}{4} = f'(1) - h[f(1) + f'(1)] \quad (66)$$

$$\frac{-C_{m_{\dot\theta}}}{4} = 2\left[f(1) - \int_0^1 xf(x)\,dx\right] - h[2f(1) + f'(1)] + h^2[f(1) + f'(1)] \quad (67)$$

The dynamic stability of the pitching motion of any thin airfoil can now be studied on the basis of Eq. (67). In particular, such a motion is always dynamically stable if $(-C_{m_{\dot\theta}})>0$ for any h. Writing Eq. (67) in the form

$$-C_{m_{\dot\theta}}/4=a-bh+ch^2 \tag{68}$$

we see that the minimum value of $-C_{m_{\dot\theta}}$ occurs when

$$\frac{\partial}{\partial h}\frac{-C_{m_{\dot\theta}}}{4}=-b+2ch=0 \tag{69}$$

or when

$$h=b/2c=h_{\text{crit}} \tag{70}$$

When Eq. (70) is substituted in Eq. (68), the minimum value of $-C_{m_{\dot\theta}}$ takes the form

$$(-C_{m_{\dot\theta}}/4)_{\min}=a-(b^2/4c) \tag{71}$$

and this quantity must be greater than zero to insure the dynamic stability of the airfoil at all values of h.

Consider now, as a subcase of the class of airfoils for which $f(x)$ is a monotonically increasing function, the set of power-law airfoil shapes defined by

$$f(x)=f(1)x^m, \qquad m>½ \tag{72}$$

Airfoils with $m<1$ are blunt-nosed with convex surfaces, while those with $m>1$ are sharp-nosed with concave surfaces. From Eq. (67), a, b, and c take the simple forms

$$a=2f(1)\left(\frac{m+1}{m+2}\right)$$
$$b=f(1)(m+2)$$
$$c=f(1)(m+1) \tag{73}$$

Substituting in Eq. (71), we have

$$\left(\frac{-C_{m_{\dot\theta}}}{4}\right)_{\min}=f(1)\left[2\left(\frac{m+1}{m+2}\right)-\frac{(m+2)^2}{4(m+1)}\right] \tag{74}$$

which is greater than zero as long as

$$m<1+\sqrt{5} \tag{75}$$

On the basis of this result we conclude that the pitching motion of any slender blunt-nosed airfoil having convex surfaces is dynamically stable at all axis positions. On the other hand, the result suggests that the pitching motion of a sharp-nosed airfoil having concave surfaces will tend toward dynamic instability over a range of axis positions if the curvature of the surfaces is increased beyond a certain value (Fig. 2). It is interesting to note that this condition of dynamic instability arises in part out of a destabilizing contribution from C_{m_α} [Eq. (57)] to the total damping-in-pitch. This unusual circumstance has its origin in the radical rearward displacement experienced by the aerodynamic center of the loading due to angle of attack as the concave curvature of the surfaces is increased.

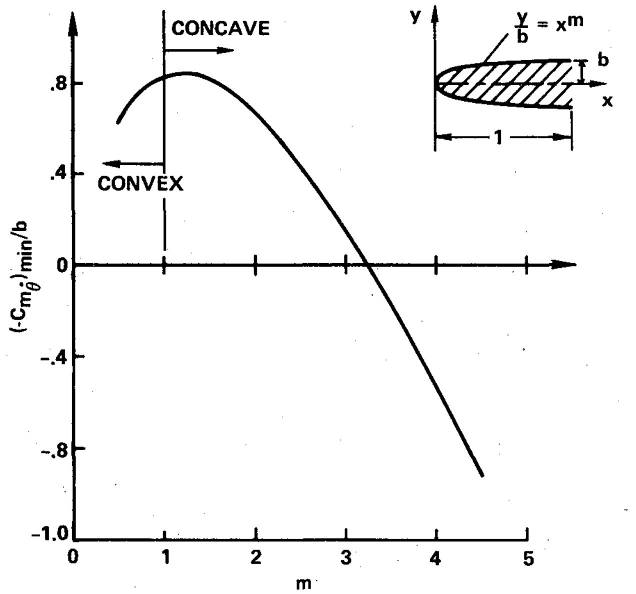

Fig. 2 Variation with power m of minimum damping-in-pitch derivative for slender power-law airfoil.

VI. Discussion and Conclusions

In this paper a theory has been developed of unsteady high Mach number flow over oscillating airfoils based on the model of Newton-Busemann; that is, the surface pressure is assumed to consist of the impact pressure plus a centrifugal correction. Like all other Newtonian flow theories it is, of course, subject to the usual restriction that it does not apply beyond the steady Newtonian flow separation point where

$$\mu^2(x)f'^2(x)+\kappa(x)\int_0^x \mu(\xi)f'(\xi)\mathrm{d}\xi=0 \tag{76}$$

The theory developed in this paper is otherwise general in that it applies for an airfoil of arbitrary thickness, arbitrary shape with a sharp or blunt leading edge, which oscillates about any axis position. For the case of an oscillating wedge, it gives results identical with gasdynamic theory in the double limit as $\gamma\to 1$ and $M_\infty\to\infty$, independently.

It is proved in this paper that the pitching motions of thin airfoils having blunt noses and convex surfaces, and of wedges of arbitrary thickness, are always dynamically stable according to Newton-Busemann theory. On the other hand, thin airfoils having sharp noses and concave surfaces are found to tend toward dynamic instability over a range of axis positions if the curvature of the surfaces exceeds a certain limit.

As a byproduct, it is also shown that the paper recently published by Barron and Mandl[8] on Newtonian flow over oscillating power-law bodies is erroneous and that their conclusion regarding the dynamic stability of these bodies is misleading.

Appendix: An Alternative Method for Deriving $P_1(x)$

In this appendix, we derive an expression for $P_1(x)$ [the coefficient of θ in Eq. (50)] by steady flow considerations alone. To this end, we let $h=0$, $Y_c(t)=0$, and $\theta=$const, hence $\gamma(t)=\dot\gamma(t)=\dot\theta(t)=0$. Then the body-fixed system of coordinates $0xy$ differs from the flow-fixed system of coordinates $0XY$ only by a rotation about the origin 0 through a small angle θ (Fig. 1).

Consider an airfoil, the surface of which is described by $y=f(x)$ referred to the body-fixed system of coordinates. When the airfoil is placed at a small angle of pitch θ, the equation of its surface in flow-fixed coordinates becomes§

$$Y=f(X)+\theta\{X+f(X)f'(X)\}\equiv F(X) \tag{A1}$$

§Consistent with the main text, terms of $O(\theta^2)$ are neglected throughout.

Accordingly, the steady pressure at a small angle of pitch θ on the airfoil, whose surface equation is $y=f(x)$, is the same as the steady pressure at $\theta=0$ on the airfoil whose surface equation is $Y=F(X)$.

Now the formula for steady Newton-Busemann pressure on an airfoil $Y=F(X)$ at $\theta=0$ can be found in any standard textbook on hypersonic flow (e.g., Ref. 9, p. 123). It is given by the expression for $P_0(x)$ in Eq. (51) when x and $f(x)$ are replaced respectively by X and $F(X)$. Thus,

$$p=\mu^2(X)F'^2(X)+\kappa(X)\int_0^X \mu(\xi)F'(\xi)\mathrm{d}\xi \tag{A2}$$

where

$$\mu(X)=[1+F'^2(X)]^{-1/2} \tag{A3}$$

$$\kappa(X)=F''(X)/[1+F'^2(X)]^{3/2} \tag{A4}$$

Substituting Eq. (A1) for F in Eq. (A2) and expanding the resulting expression in a power series in θ, we get

$$p=P_0(X)+\theta P_1^*(X) \tag{A5}$$

where

$$P_1^*(X)=\mu^3\Big\{2(1+f'^2+ff'')f'\mu+f''\int_0^X(1+f'^2+ff'')\mu^3\mathrm{d}\xi$$

$$+(f'''-3f'f''^2\mu^2)f\int_0^X f'\mu\mathrm{d}\xi\Big\} \tag{A6}$$

When Eq. (A5) is expressed as a function of the body-fixed coordinate x through the transformation $X=x-\theta f(x)$, it becomes identically

$$p=P_0(x)+\theta P_1(x) \tag{A7}$$

where $P_0(x)$ and $P_1(x)$ are the expressions given in Eq. (51) of the text. Therefore, we conclude that $P_1(x)$, the coefficient of θ in Eq. (50), is derivable by an alternative method which requires knowledge only of the result from steady Newton-Busemann theory. Since the latter theory has been proved[4] to be equivalent to limiting gasdynamic theory, it follows that our $P_1(x)$ must be identical to the expression that would result from limiting gasdynamic theory. This means that, to the first order in frequency, a correct limiting gasdynamic theory must yield $P_1(x)$ in body-fixed coordinates, or, what amounts to the same thing, Eq. (A6) in flow-fixed coordinates, for the part of the pressure proportional to θ.

In the paper of Barron and Mandl,[8] the flow-fixed coordinate is used (their x is our X), and their result for pressure is in the same form as our Eq. (A5). But their coefficient of θ [Re P in Eq. (27) of Ref. 8] is very different from our Eq. (A6). Indeed, when $\theta=$ const, hence $k=0$, their Re P vanishes identically. Now it is easy to show that P_1^* of Eq. (A6) cannot vanish identically for any body shape. Hence, the expression for Re P in Ref. 8 must be incorrect.

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